A PROBABILISTIC DYNAMIC MODEL ASSOCIATED TO A JACKSON NETWORK, WITH SERIES QUEUES; THE SOLUTION OF THE ASYMPTOTICALLY STABLE STEADY STATE

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Abstract: This article studies in detail a subclass of Jackson networks class, namely the subclass of computer networks with queues in series and comes to prove a theorem which is called the Jackson's theorem, which provides a formula that gives an analytic expression of the probability distributions for the asymptotically stable equilibrium state of this subclass.

The solution of this model in asymptotically stable equilibrium state will provide, at every moment, the probability that in the network nodes to be a certain number of transitions, in the waiting tail of the node or in processing by the server of the respectively network node.

Keywords: computer network, JACKSON network, network with series queues, server, sorting process.

1. Preliminary

The behavior of computer networks is characterized by the presence of some congestion points of transitions, called network nodes [1, 2].

In every network node forms a waiting queue where the transitions arrived in the node wait to be selected according to the discipline associated to the waiting queue, in order to be processed.

So we consider a network node as an entity made of a server or processing unit and a waiting queue, depicted as in Fig. 1[1].

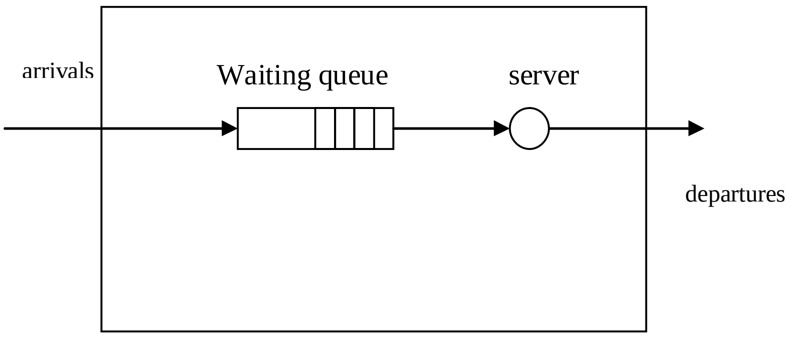

Figure. 1 *Network node*

where every transition, after has been processed by the server, is sent in the queue of another server.

The name of such a network class comes from the name of the one whome, using stochastic processes, like multi dimensional processes of birth and death, discovered the solution of mathematic model associated to the net in a simplified shape of product.

Jackson network class contains subclasses of network with series queues, with parallel queue, acyclic, with feedback and with queues in local balance.

Collection of such network nodes and interaction between them forms a computer network with waiting queues.

Definition 1.1 It’s called Jackson network a computer network with waiting queues where every server is preceded by a waiting string and

We’ll build a mathematic model associated to a Jackson type computer network, obtaining a dynamic probabilistic model.

The solution of this model will give, in every moment, the probability that in network’s nodes to be a number of transitions in the waiting queue of the node or in processing by the server of net node.

2. Theoretical results

The main result of the article requires the following result due to Burke's demonstration, whose demonstration is in [2].

Theorem 2.1. If in a net node the process of transition arrivals is Poissonian with α parameter and the process of transition processing is Poissonian with β parameter, then the process of transition departures of net node is Poisson with $\alpha+\beta$ parameter.

Theorem 2.2 We consider Σ a computer network made of N net nodes in series, to which the influx transitions is Poissonian with λ rate and the processing flow in net node i is Poissonian with rate μ_i, i=$\overline{1,N}$. If utilization factors $\rho_i = \frac{\lambda}{\mu_i} \in (0,1)$ for every i=$\overline{1,N}$, then the solution of model network in series proper to the asymptotically stable steady state is: $p(n_1,n_2,\ldots n_N) = \prod_{i=1}^{N}(1-\rho_i)\rho_i^{n_i}$.

Proof

We note $Q_i(\tau)$ the number of transitions in node i at time $\tau, i=\overline{1,N}$ [1]

Sought to determine the probability that at time τ to have n_i transitions in node i, $i=\overline{1,N}$, that is the probability:

$$P(n_1,n_2,\ldots,n_N,\tau)=P(Q_1(\tau)=n_1,Q_2(\tau)=n_2,\ldots,Q_N(\tau)=n_N)$$

In the asymptotically stable state for any $n_i \geq 0, \forall i=\overline{1,N}$ there is $\lim_{\tau\to\infty} P(n_1,n_2,\ldots,n_N,\tau)$ and is finite, value noted $p(n_1,n_2,\ldots,n_N)$.

The probability distribution $p(n_1,n_2,\ldots,n_N)$, proper to the asymptotically stable steady state, shows a description of the computer net average behavior on long-term.

We note $S_{n_1,n_2,\ldots,n_N}$ the network state proper to the case in which, for $\forall i=\overline{1,N}$, in node i there are n_i transitions.

Thus $p(n_1,n_2,\ldots,n_N) = P(S_{n_1,n_2,\ldots,n_N})$ that is the probability distribution of the asymptotically stable steady state represents the probability that the network be in state $S_{n_1,n_2,\ldots,n_N}$. Will obtain these probabilities considering the mathematical model associated to the network as a stochastic process type birth and death multi-dimensional.

Next, we demonstrate the assertion from enunciates through induction after N(N≥2).

Case N=2.

Consider a JACKSON network with two series queues, as in Fig.2 in which the transactions arrived at the end of queue 1,after a Poisson flow of rate λ, waits to be processed by server of node 1. After a transaction is processed by this server, will go at the end of the queue of node 2, where waits to be processed by this node's server, while server 1 selects a new transaction from it's queue, according to the order chosen for the queue. As per BURKE theorem, the arrivals to queue of node 2 are also Poisson of parameter $\lambda + \mu_1$, where μ_1 is the rate o server's 1 processing. Thereby the arrivals to queue of node 2 are a Poisson flow of rate $\lambda + \mu_1$. Because the rate of transitions processing by server 2 is μ_2 according to BURKE theorem, transitions that leave this network are a Poisson flow of rate :
$\lambda + \mu_1 + \mu_2$.

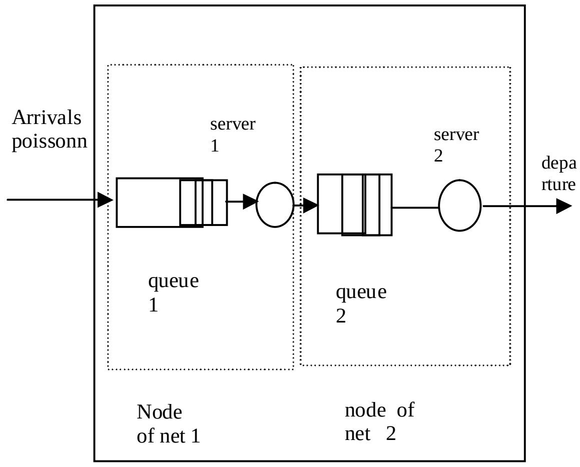

Figure. 2.

Follow to determine the probability that at time τ to have n_1 transitions in node 1 and n_2 transitions in node 2, that is:

$$P(n_1,n_2,\tau) = P(Q_1(\tau) = n_1 \wedge Q_2(\tau) = n_2).$$

Therefore, we will calculate successively the next probabilities:

$P(0,0,\tau + \Delta\tau)$ = probability of not having transitions in any node at time $\tau+\Delta\tau$.

$P(0,n_2,\tau + \Delta\tau)$ =probability of not having only in node 2, n_2 transitions at time $\tau+\Delta\tau(n_2\geq1)$.

$P(n_1,0,\tau + \Delta\tau)$ = probability of having only in node 1, n_1 transitions at time $\tau+\Delta\tau(n_1\geq1)$.

$P(n_1,n_2,\tau + \Delta\tau)$ = probability of having n_1 transitions in node 1 and n_2 transitions in node 2 at time $\tau+\Delta\tau$ $(n_1 \geq 1$ şi $n_2 \geq 1)$.

Using the hypothesis of a stochastic process type birth death, we obtain the following:

$P(0,0,\tau + \Delta\tau) = P(0,0,\tau)\cdot(1- \lambda\Delta\tau) + P(0,1,\tau)\mu_2 \cdot \Delta\tau\,(1- \lambda\Delta\tau) + 0(\Delta\tau)$

$\Rightarrow P(0,0,\tau + \Delta\tau) = P(0,0,\tau) - \lambda P(0,0,\tau)\Delta\tau + \mu_2 P(0,1,\tau)\Delta\tau + 0(\Delta\tau) \Rightarrow$

$$\Rightarrow \frac{P(0,0,\tau+\Delta\tau)-P(0,0,\tau)}{\Delta\tau} =$$

$$= -\lambda P(0,0,\tau) + \mu_2 P(0,1,\tau) + \frac{0(\Delta\tau)}{\Delta\tau}.$$

Passing to limit after $\Delta\tau \to 0$ obtain:

$$\frac{d}{d\tau}P(0,0,\tau) = -\lambda P(0,0,\tau) + \mu_2 P(0,1,\tau)$$

$P(0,n_2,\tau+\Delta\tau) =$
$=P(0,n_2,\tau)[1-\lambda\Delta\tau][1-\mu_2\Delta\tau]+P(1,n_2-1,\tau)\cdot\mu_1\Delta\tau[1-$
$-\lambda\Delta\tau)[1-\mu_2\Delta\tau]+P(0,n_2+1,\tau)\cdot\mu_2\Delta\tau[1-\lambda\Delta\tau)+0(\Delta\tau)$.

$$\Rightarrow \frac{P(0,n_2,\tau+\Delta\tau)-P(0,n_2,\tau)}{\Delta\tau} =$$

$= -(\mu_2 + \lambda)P(0,n_2,\tau)+ \mu_1 P(1,n_2 - 1,\tau) + \mu_2 P(0,n_2+$
$+ 1,\tau)+\dfrac{0(\Delta\tau)}{\Delta\tau}$.

Passing to limit after $\Delta\tau \to 0$ obtain:

$\dfrac{d}{d\tau}P(0,n_2,\tau)=-(\lambda+\mu_2)P(0,n_2,\tau)+\mu_1 P(1,n_2-1,\tau)+$
$+\mu_2 P(0,n_2+1,\tau)$

Analogue :

$P(n_1,0,\tau+\Delta\tau)=P(n_1,0,\tau)[1-\lambda\Delta\tau][1-\mu_1\Delta\tau]+$
$+P(n_1-1,0,\tau)\cdot\lambda\Delta\tau[1-\mu_1\Delta\tau]+P(n_1,1,\tau)\cdot\mu_2\Delta\tau[1- \lambda\Delta\tau)[1-\mu_1\Delta\tau] + 0(\Delta\tau)$

$$\Rightarrow \frac{P(n_1,0,\tau+\Delta\tau)-P(n_1,0,\tau)}{\Delta\tau} =$$

$$= -(\mu_1+\lambda)P(n_1,0,\tau)+\lambda P(n_1-1,0,\tau)+\mu_2 P(n_1,1,\tau)+\frac{0(\Delta\tau)}{\Delta\tau}$$

Thus:

$\dfrac{d}{d\tau}P(n_1,0,\tau)=-(\lambda+\mu_1)P(n_1,0,\tau)+\lambda P(n_1-1,0,\tau)+$
$+\mu_2 P(n_1,1,\tau)$

Finally:

$P(n_1,n_2,\tau+\Delta\tau)=P(n_1,n_2,\tau)[1-\lambda\Delta\tau)[1-\mu_1\Delta\tau][1- \mu_2\Delta\tau]+$
$+P(n_1 - 1,n_2,\tau)\cdot\lambda\Delta\tau\cdot[1- \mu_1\Delta\tau][1- \mu_2\Delta\tau] +$
$+P(n_1,n_2 +1,\tau)\cdot[1 - \lambda\Delta\tau)[1 - \mu_1\Delta\tau]\,\mu_2\Delta\tau +$
$+P(n_1+1,n_2 -1,\tau)\cdot[1- \lambda\Delta\tau)\cdot\mu_1\Delta\tau\cdot(1 - \mu_2\Delta\tau) + 0(\Delta\tau)\Rightarrow$

$$\Rightarrow \frac{P(n_1,n_2,\tau+\Delta\tau)-P(n_1,n_2,\tau)}{\Delta\tau} =$$

$=-(\mu_1+\mu_2+\lambda)\cdot P(n_1,n_2,\tau)+\lambda P(n_1-1,n_2,\tau)+$
$+\mu_2\cdot P(n_1,n_2+1,\tau)+\mu_1 P(n_1+1,n_2 -1,\tau)+\dfrac{0(\Delta\tau)}{\Delta\tau}$

Passing to limit after $\Delta\tau \to 0$ obtain:

$\dfrac{d}{d\tau}P(n_1,n_2,\tau) = -(\lambda + \mu_1 + \mu_2)\cdot P(n_1,n_2,\tau) +$
$+\lambda P(n_1-1,n_2,\tau) + \mu_1 P(n_1 +1,n_2 -2,\tau) + \mu_2\cdot P(n_1,n_2 +1,\tau)$

Conclusively, for the model of computer network with two series nodes, is obtained the following system of differential equation:

$$\frac{d}{d\tau}P(0,0,\tau) = -\lambda P(0,0,\tau) + \mu_2 P(0,1,\tau)$$

$$\frac{d}{d\tau}P(0,n_2,\tau) =$$
$$= -(\lambda+\mu_2)P(0,n_2,\tau) + \mu_1 P(1,n_2-1,\tau) + \mu_2 P(0,n_2+1,\tau)$$

$$\frac{d}{d\tau}P(n_1 0,\tau) =$$
$$= -(\lambda+\mu_1)P(n_1,0,\tau) + \lambda P(n_1-1,0,\tau) + \mu_2 P(n_1,1,\tau)$$

$$\frac{d}{d\tau}P(n_1,n_2,\tau) = -(\lambda+\mu_1+\mu_2)\cdot P(n_1,n_2,\tau) +$$
$$+\lambda P(n_1-1,n_2,\tau) + \mu_1 P(n_1+1,n_2-1,\tau) + \mu_2 P(n_1,n_2+1,\tau)$$

$\sum_{n_1,n_2\ge 0} P(n_1,n_2,\tau) = 1,\quad \forall\tau > 0$ (complimentarily condition)

In general case, this system is extremely difficult to resolve. In the stabile asymptotically balance is known that for any $n_1,n_2 \ge 0$ there is, and id finite :

$$\lim_{\tau\to\infty} P(n_1,n_2,\tau) = p(n_1,n_2)$$

The probability distribution $p(n_1,n_2)$ of stabile asymptotically balance delivers a description of the average behavior of the computer network long-term. In this condition, the system above becomes:

$$\begin{cases} -\lambda p(0,0) + \mu_2 p(0,1) = 0 \\ -(\lambda+\mu_2)p(0,n_2) + \mu_1 p(1,n_2-1) + \mu_2 p(0,n_2+1) = 0 \\ -(\lambda+\mu_1)p(n_1,0) + \lambda p(n_1-1,0) + \mu_2 p(n_1,1) = 0 \\ -(\lambda+\mu_1+\mu_2)p(n_1,n_2) + \lambda p(n_1-1,n_2) + \\ +\mu_1 p(n_1+1,n_2-1) + \mu_2 p(n_1,n_2+1) = 0 \\ \sum_{n_1,n_2\ge 0} p(n_1,n_2) = 1 \end{cases}$$

We consider:

$p(n_1,n_2) = 0,\ \forall n_1 < 0$ or $\forall n_2 < 0$
$\mu_1 p(0,n_2) = 0$
$\mu_2 p(n_1,0) = 0.$

relations that are natural if we think of their interpretation.

Thus $\mu_1 p(0,n_2)$ is the probability of processing a transition in node 1, but in this node we have no transition, and so $\mu_1 p(0,n_2) = 0$.

Analogue $\mu_2 p(n_1,0) = 0$.

In these hypothesis, the system of the first 4 equations proper to the state of stabile asymptotically balance is reduced to a single equation[1, 2], which is:

$\lambda p(n_1 -1,n_2) + \mu_1 p(n_1 +1,n_2 -1) + \mu_2 p(n_1,n_2 +1)= = (\lambda + \mu_1 + \mu_2)p(n_1,n_2)$

From this relation, deduce that in case of stabile asymptotically balance, the flow of each state is conserved, that is for any state, the input flow coincides with output flow of this state.

To this relation adds the obvious relation $\sum_{n_1,n_2\geq 0} p(n_1,n_2)=1$.

If note S_{n_1,n_2} the network state appropriate to the case in which in node 1 there are n_1 transitions and in node 2 there are n_2 transitions, in the relation above see that the states to which and from which have the transitions with state S_{n_1,n_2} are:

$S_{n_1+1,n_2-1}, S_{n_1-1,n_2}, S_{n_1,n_2+1}$ as inputs of state S_{n_1,n_2}

$S_{n_1+1,n_2}, S_{n_1-1,n_2+1}, S_{n_1,n_2-1}$ as outputs for S_{n_1,n_2}.

The interaction of these states is given by diagram in Fig. 3.

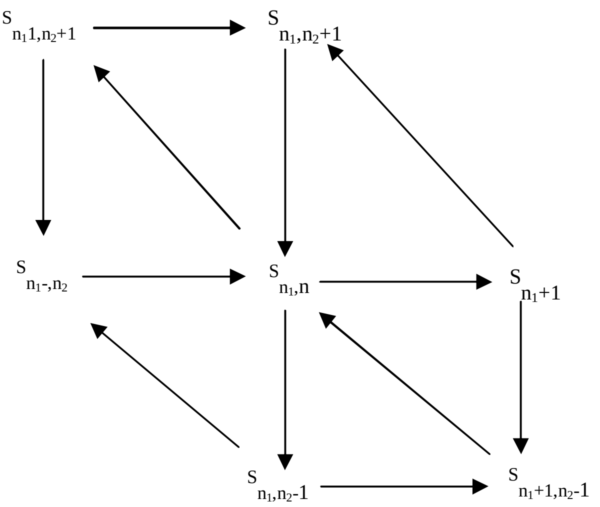

Fig 3

Because in any system in stabile balance, the flow rate of input in a state is the same with the flow rate of output from that state, and that is for any state of the system, results that in our case, by writing this relation of flow's preservation for state S_{n_1,n_2}, obtain:

Flow rate of input for :

S_{n_1,n_2} $=\lambda p(n_1-1,n_2)+\mu_1 p(n_1+1,n_2-)+\mu_2 p(n_1,n_2+1)$.

Flow rate of output from :

S_{n_1,n_2} $= (\lambda + \mu_1 + \mu_2)\, p(n_1,n_2)$.

By condition of flow's preservation in S_{n_1,n_2} obtain relation:

$\lambda\, p(n_1-1,n_2) + \mu_1\, p(n_1 + 1,n_2-1) + \mu_2\, p(n_1,n_2 + 1)= = (\lambda + \mu_1 + \mu_2)\, p(n_1, n_2)$

which is actually the relation equivalent to the probability system in case of stabile asymptotically balance.

Notice that if we put into a rectangular network n_1On_2, the neighbor states for S_{n_1,n_2} ,then every flow rate is well targeted, which is: $\underrightarrow{\lambda}$ from left to right, ↓ μ_2 downwards and ↖µ1 direction SE-NV.

Since in the considered computer network with series queues, the two net nodes are independent, Jackson [1] comes with the idea of looking for solutions to the system corresponding to stabile asymptotic balance as a product, that is $p(n_1,n_2)=p_1(n_1)\cdot p_2(n_2)$, where $p_i(n_i)$ is the probability of having in node i, n_i transitions in state of stabile asymptotic balance and so:

$$\sum_{n_1\geq 0} p_1(n_1)=1 \; si \; \sum_{n_2\geq 0} p_2(n_2)=1 \qquad (*)$$

With this choice, the equation above becomes:

$\mu_1 p_1(n_1 + 1)\, p_2(n_2 - 1) + \lambda p_1(n_1 - 1)\, p_2(n_2) + +\mu_2 p_1(n_1)\, p_2(n_2 +1)=(\lambda+\mu_1 + \mu_2)\, p_1(n_1)p_2(n_2)$ (1)

For $n_1 = n_2 = 0$ and using the hypothesis above, obtain the equation:

$\mu_1 p_1(1)p_2(-1) + \lambda p_1(-1)p_2(0) + \mu_2 p_1(0)p_2(1) =$
$= (\lambda + \mu_1 + \mu_2)\, p_1(0)p_2(0) \Leftrightarrow$
$\mu_2 p_1(0)p_2(1) = \lambda p_1(0)p_2(0) \Leftrightarrow$
$p_1(0)[\mu_2 p_2(1) - \lambda p_2(0)] = 0.$

We have the following situation possible:

1.If $p_1(0) = 0$ in relation (1) above $n_1 = 0$ obtain:

$\mu_1 p_1(1)p_2(n_2 - 1)+\lambda p_1(-1)p_2(n_2)+\mu_2 p_1(0)p_2(n_2 +1) = (\lambda + \mu_1 + \mu_2)\, p_1(0)p_2(n_2)$

Using the hypothesis above and $p_1(0)=0$ obtain: $\mu_1 p_1(1) \cdot p_2(n_2-1) = 0$

If $p_1(1) = 0$ is demonstrated by induction that $p_1(n_1) = 0$ for any $n_1 \geq 0$.

If $p_2(n_2 - 1) = 0$ results that $p_2(n_2) = 0$ for any $n_2 \geq 0$.

Both situations are false, because they contradict relations (*) above.

2.If $\mu_2 p_2(1) - \lambda p_2(0)=0 \Rightarrow p_2(1)=\dfrac{\lambda}{\mu_2} p_2(0)$ (2)

With this value go in relation (1) after $n_2 = 0$.

Have: $\mu_1 p_1(n_1 + 1)p_2(-1) + \lambda p_1(n_1 - 1)p_2(0) + +\mu_2 p_1(n_1)p_2(1) = (\lambda + \mu_1 + \mu_2)\, p_1(n_1)p_2(0)$

Results: $\lambda p_1(n_1 - 1)p_2(0) + \mu_2 p_1(n_1)p_2(1) =$
$=(\lambda + \mu_1)\, p_1(n_1)p_2(0)$

Replacing $p_2(1)$ with it's value from relation (2) obtain:

$p_2(0)[\lambda p_1(n_1-1)-\lambda+\mu_1)p_1(n_1)]+\mu_2 p_1(n_1)\dfrac{\lambda}{\mu_2}p_2(0)=0 \Rightarrow$

$\Rightarrow p_2(0)[\lambda p_1(n_1-1)-\mu p_1(n_1)]=0.$

If $p_2(0) = 0$ is shown that above are obtained null values for p_1 and p_2, which is false.

Thus, results that for $p_1(n_1) = \dfrac{\lambda}{\mu_1} p_1(n_1 - 1)$ for any $n_1 \geq 1$.

Writing this relation as : $\dfrac{p_1(k)}{p_1(k-1)} = \dfrac{\lambda}{\mu_1}$, $k \geq 1$

and doing the product:

$\prod_{k=1}^{n_1}\dfrac{p_1(k)}{p_1(k-1)} = \left(\dfrac{\lambda}{\mu_1}\right)^{n_1}$ obtain $\dfrac{p_1(n_1)}{p_1(0)} = \left(\dfrac{\lambda}{\mu_1}\right)^{n_1}$

and so $p_1(n_1) = \left(\dfrac{\lambda}{\mu_1}\right)^{n_1} \cdot p_1(0)$ (**)

From relation: $\sum_{n_1\geq 0} p_1(n_1) = 1 \Rightarrow$

$$\sum_{n_1\geq 0}\left(\frac{\lambda}{\mu_1}\right)^{n_1} p_1(0) = 1 \Rightarrow p_1(0) = \frac{1}{\sum_{n_1\geq 0}\left(\frac{\lambda}{\mu_1}\right)^{n_1}} = 1 - \frac{\lambda}{\mu_1},$$

if $\dfrac{\lambda}{\mu_1} < 1$, so that the series is convergence.

Note $\rho_1 = \dfrac{\lambda}{\mu_1}$ and call ρ_1 utilization factor in net node 1.

So for $\rho_1 \in (0,1)$ have $p_1(n) = (1-\rho_1)\rho_1^{\,n}$, $\forall\, n \geq 0$,(3)

Use this result in relation (1) after $n_1 = 0$.

Obtain:

$\mu_1 p_1(1)p_2(n_2-1)+\lambda p_1(-1)p_2(n_2)+ \mu_2 p_1(0)p_2(n_2+1)=$
$=(\lambda+\mu_1+\mu_2)\, p_1(0)p_2(n_2).$

From hypothesis result:

$\mu_1 p_1(1)p_2(n_2-1)+\mu_2 p_1(0)p_2(n_2+1)-(\lambda+\mu_2)p_1(0)p_2(n_2) = 0$

Replacing p_1 with value found in relation (**) obtain:

$\mu_1\dfrac{\lambda}{\mu_1}p_1(0)p_2(n_2-1)+\mu_2 p_1(0)p_2(n_2+1)-$

$-(\lambda+\mu_2)\, p_1(0)p_2(n_2) = 0$

$\Rightarrow p_1(0)[\lambda p_2(n_2-1)+\mu_2 p_2(n_2+1)-(\lambda+\mu_2)] = 0$

We've seen above that $p_1(0) \neq 0$ and so:

$\lambda[p_2(n_2-1)-p_2(n_2)] = \mu_2[p_2(n_2)-p_2(n_2+1)]$, $n_2 \geq 1$.

Writing this relation as:

$\lambda[p_2(k-1)-p_2(k)] = \mu_2[p_2(k)-p_2(k+1)]$, $k \geq 1$

and making the sum of k from 1 to n_2 obtain:

$$\lambda\sum_{k=1}^{n_2}[p_2(k-1)-p_2(k)] = \mu_2\sum_{k=1}^{n_2}[p_2(k)-p_2(k+1)],$$

relation equivalent with:

$\lambda[p_2(0)-p_2(n_2)] = \mu_2[p_2(1)-p_2(n_2+1)]$

But from (2) $p_2(1) = \dfrac{\lambda}{\mu_2}p_2(0)$. Replacing this above and reducing the term $\lambda p_2(0)$ obtain:

$$p_2(n_2+1) = \frac{\lambda}{\mu_2}p_2(n_2)$$

In doing the above we obtain successively:

$$\frac{p_2(k+1)}{p_2(k)} = \frac{\lambda}{\mu_2}, k \geq 0 \Rightarrow \prod_{k=0}^{n_2-1}\frac{p_2(k+1)}{p_2(k)} = \left(\frac{\lambda}{\mu_2}\right)^{n_2}$$

$$\Rightarrow \frac{p_2(n_2)}{p_2(0)} = \left(\frac{\lambda}{\mu_2}\right)^{n_2} \Rightarrow p_2(n_2) = \left(\frac{\lambda}{\mu_2}\right)^{n_2} p_2(0)$$

from

$$\sum_{n_2\geq 0} p_2(n_2) = 1 \Rightarrow \sum_{n_2\geq 0}\left(\frac{\lambda}{\mu_2}\right)^{n_2} p_2(0) = 1 \Rightarrow$$

$$\Rightarrow p_2(0) = \frac{1}{\sum_{n_2\geq 0}\left(\frac{\lambda}{\mu_2}\right)^{n_2}} = 1 - \frac{\lambda}{\mu_2},$$

if $\dfrac{\lambda}{\mu_2} < 1$ so that the series to be convergence.

Note $\rho_2 = \dfrac{\lambda}{\mu_2}$ and call ρ_2 factor of utilization in node 2.

Thus for $\rho_2 \in (0,1)$ obtain:

$$p_2(n) = (1-\rho_2)\cdot\rho_2^{\,n},\ \forall n \geq 0. \qquad (4)$$

Form relations (3), (4) and from :

$p(n_1, n_2) = p_1(n_1)\cdot p_2(n_2)$ obtain the solution for the state of stabile asymptotic balance of the model as being:

$$p(n_1,n_2) = \rho_1^{n_1}\rho_2^{n_2}(1-\rho_1)(1-\rho_2), \forall n_1, n_2 \geq 0,$$

where $\rho_1 = \dfrac{\lambda}{\mu_1} \in (0,1)$, $\rho_2 = \dfrac{\lambda}{\mu_2} \in (0,1)$.

Case N>2.

Assuming now that the relation form the statements true for $\forall k = \overline{2, N-1}$.

For k= N - 1 take the first $N-1$ series nodes and form the net Σ_1 for which we know the solution:

$p(n_1,n_2,\ldots,n_{N-1}) = \prod_{i=1}^{N-1}(1-\rho_i)\rho_i^{n_i}$, according the hypothesis of induction.

Thus Σ is obtain connecting in series Σ_1 with net node N, as in Fig. 4.

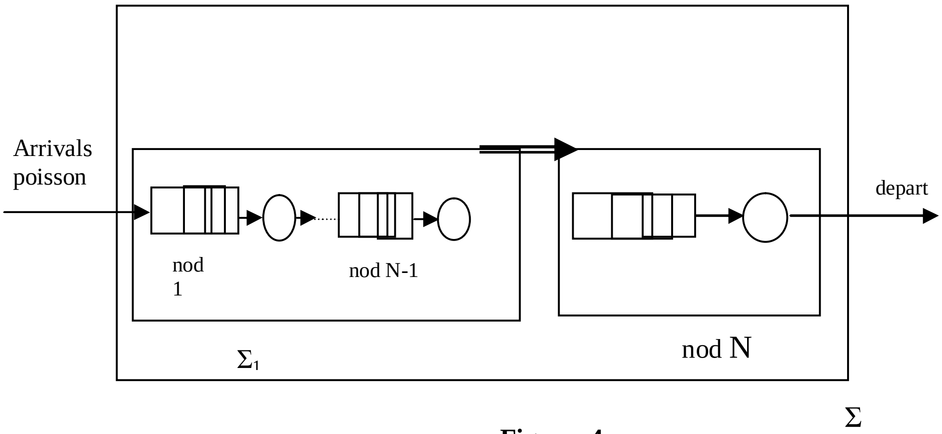

Figure. 4

Note S_{n,n_N} state corresponding to :
$n = (n_1,\ldots,n_{N-1})$ and n_N and
$p(n,n_N)= p(S_{n,n_N}) = p(S_{n_1,\ldots,n_{N-1},n_N})$ =
$=p((n_1,\ldots,n_{N-1}),n_N) = p(n_1,\ldots,n_{N-1},n_N)$.

As passing from Σ_1 in net node N is made through net node $N-1$, it can be built the next diagram of flow between states (similar to case of two series net nodes)(see Fig.5) in which
$n-1 = (n_1,\ldots,n_{N-2},n_{N-1}-1)$ and
$n+1 = (n_1,\ldots,n_{N-2},n_{N-1}+1)$

ion of flow preservation in state S_{n,n_N} and for resolving the obtained system we look for solutions like:

$$p(n,n_N) = p(n)p_N(n_N).$$

The preservation relation is:

$$\mu_{N-1}p(n+1)p_N(n_N-1) + \lambda p(n-1)p_N(n_N) + \mu_N p(n)p_N(n_N+1) = (\lambda + \mu_{N-1} + \mu_N) \cdot p(n)p_N(n_N) \quad (5)$$

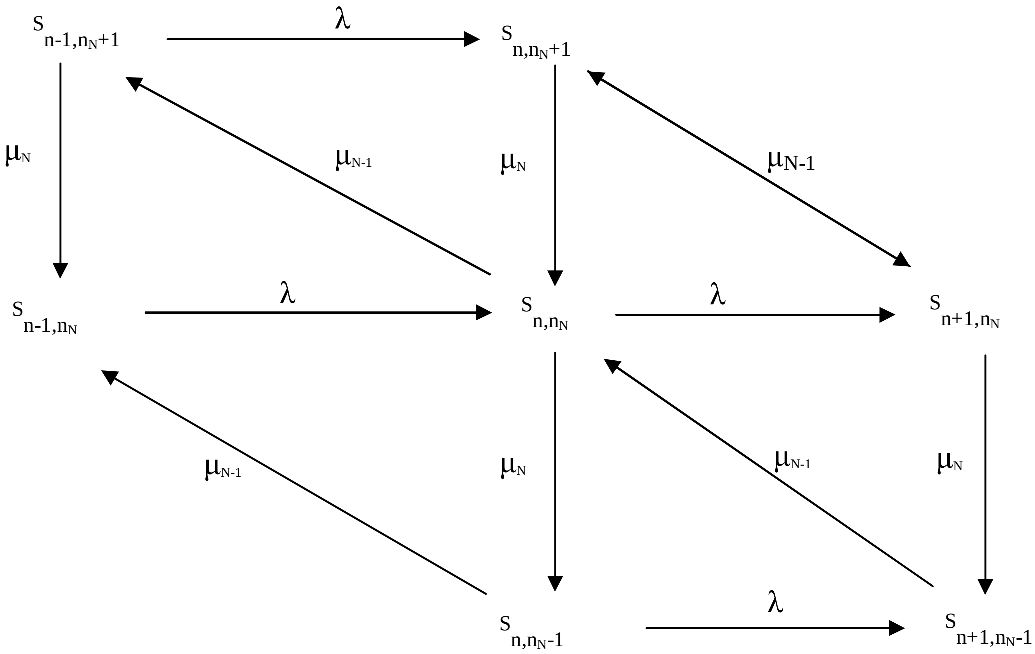

Fig. 5

Using that $p(n) = \prod_{i=1}^{N-1} (1-\rho_i)\rho_i^{n_i}$ (results from induction hypothesis) it will be obtained a relation of recurrence of order I for p_N and therefore: $p_N(n_N) = (1-\rho_N)\rho_N^{n_N}$, where $\rho_N = \frac{\lambda}{\mu_N} \in (0,1)$.

Thus : $p(n_1,\dots,n_{N-1},n_N) = p(n,n_N) = p(n)p_N(n_N) = \prod_{i=1}^{N-1}(1-\rho_i)\rho_i^{n_i} \cdot (1-\rho_N)\rho_N^{n_N} = \prod_{i=1}^{N}(1-\rho_i)\rho_i^{n_i}$.

Indeed, replacing $p(n)$ in relation (5) obtain:

$$\mu_{N-1}\prod_{i=1}^{N-2}\rho_i^{n_i}(1-\rho_i)\cdot p_{N-1}(n_{N-1}+1)\cdot p_N(n_N-1) +$$
$$+\lambda\prod_{i=1}^{N-2}\rho_i^{n_i}(1-\rho_i)\cdot p_{N-1}(n_{N-1}-1)\cdot p_N(n_N) +$$
$$+\mu_N\prod_{i=1}^{N-2}\rho_i^{n_i}(1-\rho_i)\cdot p_{N-1}(n_{N-1})\cdot p_N(n_N+1) =$$
$$=(\lambda+\mu_{N-1}+\mu_N)\prod_{i=1}^{N-2}\rho_i^{n_i}(1-\rho_i)\cdot p_{N-1}(n_{N-1})\cdot p_N(n_N)$$

We divide this relation to $\prod_{i=1}^{N-2}\rho_i^{n_i}(1-\rho_i)$ and

$$\mu_{N-1}\rho_{N-1}^{n_{N-1}+1}(1-\rho_{N-1})p_N(n_N-1) +$$
$$+\lambda\rho_{N-1}^{n_{N-1}-1}(1-\rho_{N-1})p_N(n_N) +$$
$$+\mu_N\rho_{N-1}^{n_{N-1}}(1-\rho_{N-1})p_N(n_N+1) =$$
$$=(\lambda+\mu_{N-1}+\mu_N)\rho_{N-1}^{n_{N-1}}(1-\rho_{N-1})p_N(n_N)$$

Divide through $\rho_{N-1}^{n_{N-1}-1}(1-\rho_{N-1})$ and obtain:

$$\mu_{N-1}\rho_{N-1}^2 p_N(n_N-1)+\lambda p_N(n_N) +$$
$$+\mu_N\rho_{N-1}p_N(n_N+1) =$$
$$=(\lambda+\mu_{N-1}+\mu_N)\rho_{N-1}p_N(n_N)$$

Reduce the term $\lambda p_N(n_N)$ with $\mu_{N-1}\rho_{N-1}p_N(n_N) = \mu_{N-1}\frac{\lambda}{\mu_{N-1}}p_N(n_N)$, divide with $\rho_{N-1} = \frac{\lambda}{\mu_{N-1}}$ and obtain:

$$\mu_{N-1}\rho_{N-1}p_N(n_N-1)+\mu_N p_N(n_N+1)$$
$$=(\lambda+\mu_N)p_N(n_N) \Leftrightarrow$$
$$\lambda[p_N(n_N-1)-p_N(n_N)] =$$
$$=\mu_N[p_N(n_N)-p_N(n_N+1)]$$

a relation similar to the one form which we obtained p_2. Forwards, proceed as for p_2 and finally obtain p_N as being $p_N(n_N) = (1-\rho_N)\rho_N^{n_N}$ where $\rho_N = \frac{\lambda}{\mu_N} \in (0,1)$.

In order to obtain this result, it was also needed the relation $p_N(1) = \frac{\lambda}{\mu_N}p_N(0)$ which is get if in relation (5) put $n = 0$ and $n_N = 0$.

Obtain:

$$\mu_{N-1}p(1)p_N(-1)+\lambda p(-1)p_N(0)+\mu_N p(0)p_N(1) =$$
$$=(\lambda+\mu_{N-1}+\mu_N)p(0)p_N(0) \Leftrightarrow$$
$$\Leftrightarrow p(0)[\mu_N p_N(1)-\lambda p_N(0)] = 0$$
$$\Leftrightarrow p_N(1) = \frac{\lambda}{\mu_N}p_N(0)$$

because $p(0) \neq 0$.

Observations[1].

1) Jackson has shown that forma the product of the final relationship of a mathematical model is available also in a more general case, in which are allowed transitions between states form:

$S_{n_1,\dots,n_j,\dots,n_N} \to S_{n_1,\dots,n_j+1,\dots,n_N}$, a transition enters from the environment directly to the queue of processor j.

$S_{n_1,\dots,n_i,\dots,n_N} \to S_{n_1,\dots,n_i-1,\dots,n_N}$, a transition leaves the system (computer net) through net node *i*.

$S_{n_1,\dots,n_i,\dots n_j,\dots,n_N} \to S_{n_1,\dots,n_i-1,\dots n_j+1,\dots,n_N}$,

a transition passes from net *node i* straight to *node j*.

2) Also, Jackson has shown that the final solution is obtained as a product for a more general class of networks in which is allowed processing the transitions several times in a certain net node, that is allowing cycling the transitions between entering and leaving a certain net node.

3. Conclusion

This article presents a concrete modality to model systems for the particular case of computer networks with series queues, which is the mathematical method that leads to a equations system, sometimes impossible to resolve without imposing supplementary hypothesis which are, in some cases, extremely restrictive, hypothesis that ease up the model from the real modeled system.

In this case, working in the asymptotically stable equilibrium state, the differential equations system that's obtained, impossible to resolve in general, is reduced to a resolvable equations system, which will provide, through the probability distribution $p(n_1, n_2, \dots, n_N)$, a description of the medium behaviour on long term of the computer networkwith series queues.

In the same way the other subclasses of Jackson network can be modeled, and also network type BCMP, BUZEN, etc and the theoretical results can be compared by mathematical models for performance indicators, such as: using a node, use two or more node in the same time, residence time of transactions in network, medium length of waiting tails, a node's efficiency and many other, with the results given by modeling through other methods, such as modeling trough Coloured Petri nets [3,4].

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