MODELING OF PRESSURE DISTRIBUTIONS AND NORMAL DISPLACEMENTS IN THE CASE OF LOADS DISTRIBUTED OVER SURFACES BOUND BY CLOSED CONICAL CURVES

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Abstract: *Mechanical contacts found in practice occur over finite areas, often bound by closed conical curves. In such situations it is useful to evaluate the effects of some particular load distributions. Mechanical contacts found in practice occur over finite areas, often bound by closed conical curves. In such situations it is useful to evaluate the effects of some particular loads. The present study considered the particular case where the closed conical curve is represented by an ellipse. In this situation, three particular cases were analyzed and the normal displacement was modeled mathematically.*

Keywords: *simulation, contact mechanics, pressure distribution, closed conical boundaries*

1. Introduction

Because the dimensions of the contact area are small compared to the curvature radii of the two adjacent surfaces, evaluated in the area of initial contact, it is often considered that these radii are infinite. In this case, the bodies in contact can be assimilated by elastic half-spaces. In this context, the elastic half-space represents that part of space bound by a plane, which is filled with an elastic material of known parameters ν, E and G.

The elastic half-space can be stressed by loads applied on its boundary plane. In the simplest situations, these loads can be concentrated or evenly distributed along a straight line. The problems of determining the displacements and stresses produced in the half-space by these simple loads are called fundamental problems of the elastic half-space, [Glovnea,1999]. The case of a half-space loaded by a concentrated force perpendicular to the boundary plane is called *"the Boussinesq problem".* If the concentrated force is contained within the boundary plane, the loading case is called "*the Cerruti problem*". The *"combined Boussinesq-Cerruti problem"*, appears when the half space is loaded by a randomly oriented concentrated force applied in a point from the half-space boundary plane. In the situation where the elastic half-space is stressed by a force uniformly distributed along a line contained within the boundary plane, the resulting case is called *"the Flamant problem"*. In the same category, of fundamental problems of the elastic half-space, falls the principle of overlapping effects, which allows the generalization of abovementioned problems in the case of continuously distributed loads over a certain region of the half-space bordering plane.

Mechanical contacts found in practice occur over finite areas, often bound by closed conical curves. In such situations it is useful to evaluate the effects of some particular loads.

The present study considered the particular case where the closed conical curve is represented by an ellipse. In this situation, three particular cases were analyzed and the normal displacement was modeled mathematically.

2. Load distribution over an elliptical area

In the case of loading areas bound by an ellipse having a and b as half-axes, the coordinate system origin is chosen in the ellipse center. According to the method developed by Johnson, [Johnson,1985], the load distribution is assumed to be described by Eq. (1):

$$p(x,y)=p_0\left[1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right]^n, \qquad (1)$$

where p_0 represents the central pressure and n is an exponent that can take various values.

In order to estimate the effects of the considered pressure distribution, the principle of overlapping effects is applied to Boussinesq's problem for the elastic half-space.

The displacement along the z axis of an arbitrary point of the boundary plane (initially at z=0) can be expressed by:

$$w(x,y)=\frac{1-\nu^2}{\pi E}p_0\cdot\iint_A\left[1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right]^n\cdot \cdot\left[(x-x')^2+(y-y')^2\right]^{-\frac{1}{2}}dx'dy' \qquad (2)$$

Eq. (2) is typical for a single layer potential, having as source density the pressure distribution. The integral over the elliptic area, A, can be customized from the general expression of the ellipsoid's potential in ellipsoidal coordinates, [Johnson,1985], as:

$$I(x,y,z)=\frac{\Gamma(n+1)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)}ab\cdot \cdot\int_{s_1}^{\infty}\left(1-\frac{x^2}{a^2+s}-\frac{y^2}{b^2+s}-\frac{z^2}{c^2+s}\right)^{n+\frac{1}{2}}\cdot \cdot\frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}, \qquad (3)$$

where s represents the maximum root of Eq. (4).

$$\frac{x^2}{a^2+s}+\frac{y^2}{b^2+s}+\frac{z^2}{s}-1=0 \qquad (4)$$

In the case of a flattened ellipsoid in the z plane (z=0), case in which $c\to 0$, the double integral over the contour A, Eq. (3) can be written as:

$$I(x,y,z)=\frac{\Gamma(n+1)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)}ab\cdot \cdot\int_{s_1}^{\infty}\left(1-\frac{x^2}{a^2+s}-\frac{y^2}{b^2+s}\right)^{n+\frac{1}{2}}\cdot\frac{ds}{\sqrt{(a^2+s)(b^2+s)}}, \qquad (5)$$

3. Modeling of particular cases of pressure distributions

For the present study, three particular situations of the n exponent from Eq. (1) were modeled: $n=-\frac{1}{2}$, $n=0$ and $n=\frac{1}{2}$.

3.1 Pressure distribution case I

In the first considered case, for $n=-\frac{1}{2}$, the pressure distribution, given by Eq.(1), becomes:

$$p(x,y)=\frac{p_0}{\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}}\,. \qquad (6)$$

This corresponds to a pressure tending to infinity over the elliptic contour, and to a minimum equal to p_0, in the ellipse center.

Inside the area delimited by the ellipse, the displacement *w* will become:

$$w=\frac{1-\nu^2}{E}p_0\,ab\int_0^{\infty}\frac{ds}{\sqrt{s\left(a^2+s\right)\left(b^2+s\right)}}\,. \qquad (7)$$

The relation for displacement given by Eq. (7), clearly illustrated that the displacement is constant and the surface deformed profile remains flat.

The complete elliptic integral, usually denoted by K(e), which intervenes in Eq. (7), leads to the following expression of the displacement w over the loaded area, [Johnson,1985]:

$$w(x,y)=\frac{2(1-\nu^2)}{E}p_0\,b\,K(e)\,, \qquad (8)$$

where *e* represents the eccentricity of the ellipse, $e=\sqrt{1-\beta^2}$, and $\beta=b/a$ is the ellipse aspect ratio.

3.2 Pressure distribution case II

In the case of an uniformly distributed pressure over the ellipse, $p=p_0$, it results that $n=0$. This leads to a normal displacement of the elastic half-space boundary plane points given by:

$$w(x,y)=\frac{1-\nu^2}{\pi E}p_0\cdot \iint_A\frac{dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}}\,. \qquad (9)$$

The double integral form Eq. (9) can be transformed into a simple integral if an arbitrary string is traced through the $M(x,y)$ point from the elliptic surface, inclined with an angle θ by report to the abscissa. By transformation into polar coordinates, with the radius measured along the string length $\ell(\theta)$ and considering the polar angle θ, the following integral expression of the displacement $w(x,y)$ can be obtained:

$$w(x,y)=\frac{1-\nu^2}{\pi}\frac{p_0}{E}\int_0^{\pi}\ell\,d\theta=2\beta\frac{1-\nu^2}{\pi}\frac{p_0}{E}\cdot \int_0^{\pi}\frac{\sqrt{\sin^2\theta+\beta^2\cos^2\theta-\left(\frac{x}{a}\sin\theta-\beta\frac{y}{a}\cos\theta\right)^2}}{\sin^2\theta+\beta^2\cos^2\theta}\,d\theta. \qquad (10)$$

The integral with regards to θ from Eq. (10) can't be expressed by combinations of elementary functions, so it is necessary to express it numerically. Solutions for determining the w displacement in this case, obtained via numerical integration by aid of MathCad, were presented by Diaconescu and Glovnea, [Diaconescu,1994], [Glovnea,1999].

3.3 Pressure distribution case III

In the third considered case, for $n=\frac{1}{2}$, the pressure distribution, given by Eq.(1), takes the shape of demi-ellipsoid being null along the elliptic contour and reaching maximum value p0 over the ellipse center, as yielded by Eq. (11):

$$p(x,y)=p_0\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}\,. \qquad (11)$$

The pressure distribution given in Eq.(11) is known in literature as *Hertz pressure distribution*.

The displacement w inside the elliptic area is in this case given by:

$$w(x,y)=\frac{1-\nu^2}{2E}\,\mathrm{p}_0\;ab\int_0^{\infty}\left(1-\frac{x^2}{a^2+s}-\right.$$
$$\left.-\frac{y^2}{b^2+s}\right)\frac{ds}{\sqrt{s(a^2+s)(b^2+s)}}\quad . \tag{12}$$

The expression given by Eq. (12) can be rewritten as a second degree polynomial, as showed by Johnson, in [Johnson,1985]:

$$\mathrm{w(x,y)}=\frac{1-\nu^2}{\mathrm{E}}\left(\mathrm{L}-\mathrm{Mx}^2-\mathrm{Ny}^2\right), \tag{13}$$

which denotes that the deformed surface from the interior of the elliptical contour is shaped as an elliptic paraboloid.

In Eq. (13), the L, M and N coefficients have the following expressions:

$$L=\frac{1}{2}\mathrm{p}_0\;ab\int_0^{\infty}\frac{ds}{\sqrt{s(a^2+s)(b^2+s)}}=$$
$$=p_0\cdot\mathrm{b}\cdot\mathrm{K}(e)\quad ; \tag{14}$$

$$M=\frac{1}{2}\mathrm{p}_0\;ab\int_0^{\infty}\frac{ds}{(a^2+s)\sqrt{s(a^2+s)(b^2+s)}}=$$
$$=\frac{p_0\,b}{e^2a^2}\left[K(e)-E(e)\right]\quad ; \tag{15}$$

$$N=\frac{1}{2}\mathrm{p}_0\;ab\int_0^{\infty}\frac{ds}{(b^2+s)\sqrt{s(a^2+s)(b^2+s)}}=$$
$$=\frac{p_0\;b}{\mathrm{e}^2a^3}\left[\frac{1}{\sqrt{1-e^2}}E(e)-K(e)\right]\quad , \tag{16}$$

where E(e) represents the second complete elliptical integral.

The maximum displacement occurs in the ellipse center and its value w_0 is given by:

$$\mathrm{w}_0=\mathrm{w}(0)=\frac{1-\nu^2}{\mathrm{E}}\,\mathrm{p}_0\;\mathrm{b\;K(e)}\,. \tag{17}$$

4. Conclusions

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Mechanical contacts found in practice occur over finite areas, often bound by closed conical curves. In such situations it is useful to evaluate the effects of some particular loads.

The present study considered the particular case where the closed conical curve is represented by an ellipse. In this situation, three particular cases were analyzed and the normal displacement was modeled mathematically.

The relations obtained in the present paper can easily be used for the particular case where the closed conical curve is represented by a circle.

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